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Global heat kernel estimates for symmetric Markov processes dominated by stable-like processes in exterior $C^{1,\eta}$ open sets

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Abstract

In this paper, we establish sharp two-sided heat kernel estimates for a large class of symmetric Markov processes in exterior $C^{1,\eta}$ open sets for all $t > 0$. The processes are symmetric pure jump Markov processes with jumping kernel intensity

$$\kappa(x,y)\psi(|x-y|)^{-1}|x-y|^{-d-\alpha}$$

where $\alpha \in (0,2)$, ψ is an increasing function on $[0,\infty)$ with $\psi(r) = 1$ on $0 < r \le 1$ and $c_1 e^{c_2 r^\beta} \le \psi(r) \le c_3 e^{c_4 r^\beta}$ on $r > 1$ for $\beta \in [0,\infty]$. A symmetric function $\kappa(x,y)$ is bounded by two positive constants and $|\kappa(x,y) - \kappa(x,x)| \le c_5 |x-y|^\rho$ for $|x-y| < 1$ and $\rho > \alpha/2$. As a corollary of our main result, we estimates sharp two-sided Green function for this process in $C^{1,\eta}$ exterior open sets.

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1 Introduction

In this paper, we study two-sided heat kernel estimates for a large class of symmetric Markov processes with jumps in exterior $C^{1,\eta}$ open sets for all $t > 0$. Discontinuous Markov processes and non-local Markovian operator have received much attention recently. The transition density $p(t,x,y)$ which describes the distribution of Discontinuous Markov process is a fundamental solution of involving infinitesimal generator and there are many studies in this areas in [1, 4, 5, 6, 14, 15]. Very recently in [3], two-sided estimates on $p(t,x,y)$ for isotropic unimodal Lévy processes with Lévy exponents having weak local scaling at infinity are established. Also, heat kernel estimates for a class of Lévy processes with Lévy measures not necessarily absolutely continuous with respect to the underlying measure are obtained by Kaleta and Sztonyk in [20].

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Since it is difficult to obtain two-sided estimates on Dirichlet heat kernel where points are near the boundary, Dirichlet heat kernel estimates are obtained recently for particular processes in [2, 7, 8, 9]. Very recently, the studies of two-sided Dirichlet heat kernel estimates are extended to a large class of symmetric Lévy processes and beyond in [12, 13, 21].

In this paper, we consider a large class of symmetric Markov processes whose jumping kernels are dominated by the kernels of stable-like processes which is discussed in [21]. Throughout this paper we assume that $\beta \in [0, \infty]$, $\alpha \in (0, 2)$, and $d \in \{1, 2, 3, \dots\}$. For two nonnegative functions f and g, the notation $f \asymp g$ means that there are positive constants c_1 and c_2 such that $c_1 g(x) \le f(x) \le c_2 g(x)$ in the common domain of definition for f and g. We will use the symbol ":=," which is read as "is defined to be."

Let ψ be an increasing function on $[0, \infty)$ where $\psi(r) = 1$ on $0 < r \le 1$, and $L_1 e^{\gamma_1 r^\beta} \le \psi(r) \le L_2 e^{\gamma_2 r^\beta}$ on $1 < r < \infty$. Here $L_1, L_2, \gamma_1, \gamma_2$ are positive constants. For any $r > 0$, we define $j(r) := r^{-d-\alpha}\psi(r)^{-1}$. Let $\kappa(x, y)$ be a positive symmetric function which is satisfying

$$L_3^{-1} \le \kappa(x, y) \le L_3, \quad x, y \in \mathbb{R}^d, \tag{1.1}$$

and for $\rho > \alpha/2$,

$$|\kappa(x, y) - \kappa(x, x)| \mathbf{1}_{\{|x-y|<1\}} \le L_4 |x - y|^\rho, \quad x, y \in \mathbb{R}^d,$$

where L_3, L_4 are positive constants. We define a symmetric measurable function J on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\}$ as

$$J(x, y) := \kappa(x, y) j(|x - y|) = \begin{cases} \kappa(x, y)|x - y|^{-d-\alpha}\psi(|x - y|)^{-1} & \text{if } \beta \in [0, \infty), \\ \kappa(x, y)|x - y|^{-d-\alpha}\mathbf{1}_{\{|x-y|\le 1\}} & \text{if } \beta = \infty. \end{cases} \tag{1.2}$$

For any $u \in L^2(\mathbb{R}^d, dx)$, we define $\mathcal{E}(u, u) := 2^{-1}\int_{\mathbb{R}^d \times \mathbb{R}^d}(u(x) - u(y))^2 J(x, y) dx dy$ and $\mathcal{D}(\mathcal{E}) := \{f \in C_c(\mathbb{R}^d) : \mathcal{E}(f, f) < \infty\}$ where $C_c(\mathbb{R}^d)$ is the space of continuous functions with compact support in $\mathbb{R}^d$ equipped with uniform topology. Let $\mathcal{E}_1(u, u) := \mathcal{E}(u, u) + \int_{\mathbb{R}^d} u(x)^2 dx$ and $\mathcal{F} := \overline{\mathcal{D}(\mathcal{E})}^{\mathcal{E}_1}$. Then by [15, Proposition 2.2], $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d, dx)$ and there is a Hunt process Y associated with this on $\mathbb{R}^d$ (see [18]).

It is shown in [21] that the Hunt process Y associated with $(\mathcal{E}, \mathcal{F})$ is a subclass of the processes considered in [6]. Therefore, Y is conservative and it has a Hölder continuous transition density $p(t, x, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ with respect to the Lebesgue measure. In [19, 22], this process is discussed and the upper bound estimates are obtained.

For any $x \in \mathbb{R}^d$, stopping time S with respect to the filtration of Y, and nonnegative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ where $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$ and $s \ge 0$, we have a Lévy system for Y :

$$\mathbb{E}_x \left[\sum_{s \le S} f(s, Y_{s-}, Y_s) \right] = \mathbb{E}_x \left[\int_0^S \left(\int_{\mathbb{R}^d} f(s, Y_s, y) J(Y_s, y) dy \right) ds \right] \tag{1.3}$$

(e.g., see [15, Appendix A]). It describes the jumps of the process Y, so the function J is called the jumping intensity kernel of Y.

For $a, b \in \mathbb{R}$, we use $\wedge$ and $\vee$ to denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For any positive constants a, b, T, we define functions $\Psi^1_{a,b,T}(t, r)$ on $(0, T] \times [0, \infty)$ as

$$\Psi^1_{a,b,T}(t,r) := \begin{cases} t^{-d/\alpha} \wedge tr^{-d-\alpha} e^{-br^\beta} & \text{if } \beta \in [0,1], \\ t^{-d/\alpha} \wedge tr^{-d-\alpha} & \text{if } \beta \in (1,\infty] \text{ with } r<1, \\ t \exp\left(-a\left(r\left(\log \frac{Tr}{t}\right)^{\frac{\beta-1}{\beta}} \wedge r^\beta\right)\right) & \text{if } \beta \in (1,\infty) \text{ with } r \geq 1, \\ (t/(Tr))^{ar} & \text{if } \beta = \infty \text{ with } r \geq 1 \end{cases} \tag{1.4}$$

and $\Psi^2_{a,T}(t, r)$ on $[T, \infty) \times (0, \infty)$ as

$$\Psi^2_{a,T}(t,r) := \begin{cases} t^{-d/\alpha} \wedge tr^{-d-\alpha} & \text{if } \beta = 0, \\ t^{-d/2} \exp\left(-a\left(r^\beta \wedge \frac{r^2}{t}\right)\right) & \text{if } \beta \in (0,1], \\ t^{-d/2} \exp\left(-a\left(r\left(1+\log^+ \frac{Tr}{t}\right)^{(\beta-1)/\beta} \wedge \frac{r^2}{t}\right)\right) & \text{if } \beta \in (1,\infty), \\ t^{-d/2} \exp\left(-a\left(r\left(1+\log^+ \frac{Tr}{t}\right) \wedge \frac{r^2}{t}\right)\right) & \text{if } \beta = \infty \end{cases} \tag{1.5}$$

where $\log^+ x = \log x \cdot \mathbf{1}_{\{x \geq 1\}} + 0 \cdot \mathbf{1}_{\{x<1\}}$.

By [15, Theorem 1.2], [6, Theorem 1.2 and Theorem 1.4] and [21, Theorem 1.1], it is known that for any $T > 0$, there are positive constants $C_1, c \geq 1$ and $\gamma = \gamma(\gamma_1, \gamma_2) \geq 1$ such that

$$c^{-1} \Psi^1_{C_1,\gamma,T}(t, |x-y|) \,\leq\, p(t,x,y) \,\leq\, c\, \Psi^1_{C_1^{-1},\gamma^{-1},T}(t,|x-y|) \tag{1.6}$$

for every $(t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ and

$$c^{-1} \Psi^2_{C_1,T}(t, |x-y|) \,\leq\, p(t,x,y) \,\leq\, c\, \Psi^2_{C_1^{-1},T}(t,|x-y|) \tag{1.7}$$

for every $(t, x, y) \in [T, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. Even though in [15, Theorem 1.2] and [6, Theorems 1.2 and 1.4] two-sided estimates for $p(t, x, y)$ are stated separately for the cases $0 < t \leq 1$ and $t \geq 1$, the constant 1 does not play any special role. Thus by the same proof, two-sided estimates for $p(t, x, y)$ hold for the case $0 < t \leq T$ and can be stated in the above way. We remark here that in [6, Theorems 1.2(2.b)] the case $|x - y| \asymp t$ is missing. One can see that (1.7) is the correct form to include the case $|x - y| \asymp t$ (cf. Proposition 3.6 below for the lower bound).

The goal of this paper is to establish the two-sided heat kernel estimates for Y in exterior $C^{1,\eta}$ open set. Recall that an open set D in $\mathbb{R}^d$ (when $d \geq 2$) is said to be $C^{1,\eta}$ open set with $\eta \in (0, 1]$ if there exist a localization radius $r_0 > 0$ and a constant $\Lambda_0 > 0$ such that for every $z \in \partial D$, there exists a $C^{1,\eta}$-function $\phi = \phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0$, $\nabla\phi(0) = (0, \dots, 0)$, $\|\nabla\phi\|_\infty \leq \Lambda_0$, $|\nabla\phi(x) - \nabla\phi(w)| \leq \Lambda_0 |x-w|^\eta$ and an orthonormal coordinate system CS_z of $z = (z_1, \cdots, z_{d-1}, z_d) =: (\widetilde{z},\, z_d)$ with origin at z such that $B(z, r_0) \cap D = \{y = (\tilde{y}, y_d) \in B(z, r_0) \text{ in } CS_z : y_d > \phi(\widetilde{y})\}$. The pair (r_0, Λ_0) will be called the $C^{1,\eta}$ characteristics of the open set D. Note that a $C^{1,\eta}$ open set D with characteristics (r_0, Λ_0) can be unbounded and disconnected.

Let Y^D be the subprocess of Y killed upon exiting D and $\tau_D := \inf\{t > 0 : Y_t \notin D\}$ be the first exit time from D. By the strong Markov property, it can easily be verified that $p_D(t, x, y) :=$

$p(t,x,y) - \mathbb{E}_x[p(t-\tau_D, Y_{\tau_D}, y); t > \tau_D]$ is the transition density of Y^D. Also, by the continuity and estimate of p, it is routine to show that $p_D(t,x,y)$ is symmetric and continuous(e.g., see the proof of Theorem 2.4 in [17]).

In [21, Theorem 1.2], the Dirichlet heat kernel estimates for Y^D is obtained. For the lower bound estimates on $p_D(t,x,y)$ when $\beta \in (1,\infty]$, we need the following assumption on D: *the path distance in each connected component of D is comparable to the Euclidean distance with characteristic* λ_1, i.e., for every x and y in the same component of D there is a rectifiable curve l in D which connects x to y such that the length of l is less than or equal to $\lambda_1|x-y|$. Clearly, such a property holds for all bounded $C^{1,\eta}$ open sets, $C^{1,\eta}$ open sets with compact complements, and connected open sets above graphs of $C^{1,\eta}$ functions.

Here is the main result of [21]. We denote by $\delta_D(x)$ the Euclidean distance between x and D^c.

Theorem 1.1 [21, Theorem 1.2] *Let J be the symmetric function defined in* (1.2) *and Y be the symmetric pure jump Hunt process with the jumping intensity kernel J. Suppose that $T>0$ and γ is the constant in* (1.6). *For any $\eta \in (\alpha/2, 1]$, let D be a $C^{1,\eta}$ open set in $\mathbb{R}^d$ with $C^{1,\eta}$ characteristics (r_0, Λ_0). Then the transition density $p_D(t,x,y)$ of Y^D has the following estimates.*

(1) *There are positive constants $c, C_2 \ge 1$ such that for any $(t,x,y) \in (0,T] \times D \times D$, we have*

$$c \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \Psi^1_{C_2^{-1},\gamma^{-1},T}(t, |x-y|/6) \ge p_D(t,x,y)$$

$$\ge c^{-1} \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \cdot \begin{cases} t^{-d/\alpha} \wedge t|x-y|^{-d-\alpha} e^{-\gamma|x-y|^\beta} & \text{if } \beta \in [0,1], \\ t^{-d/\alpha} \wedge t|x-y|^{-d-\alpha} & \text{if } \beta \in (1,\infty] \text{ and} \\ & |x-y| \le 4/5. \end{cases}$$

(2) *Suppose in addition that the path distance in each connected component of D is comparable to the Euclidean distance with characteristic λ_1. If $\beta \in (1,\infty]$, there are positive constants $c, C_2 \ge 1$ such that for any $(t,x,y) \in (0,T] \times D \times D$ where $|x-y| \ge 4/5$ and x,y are in a same component of D, we have*

$$p_D(t,x,y) \ge c^{-1} \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \Psi^1_{C_2,\gamma,T}(t, 5|x-y|/4)$$

(3) *If $\beta \in (1,\infty)$, there is a positive constant $c \ge 1$ such that for any $(t,x,y) \in (0,T] \times D \times D$ where $|x-y| \ge 4/5$ and x,y are in different components of D, we have*

$$p_D(t,x,y) \ge c^{-1} \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \frac{t}{|x-y|^{d+\alpha}} e^{-\gamma(5|x-y|/4)^\beta}.$$

(4) *Suppose in addition that D is bounded and connected. Then there is positive constant $c \ge 1$ such that for any $(t,x,y) \in [T,\infty) \times D \times D$ we have*

$$c^{-1} e^{-t\lambda^D} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} \le p_D(t,x,y) \le c\, e^{-t\lambda^D} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2},$$

where $-\lambda^D < 0$ is the largest eigenvalue of the generator of Y^D.

Theorem 1.1(1)–(3) give us the Dirichlet heat kernel estimates for the small time. However the large time estimates are established only for the bounded and connected $C^{1,\eta}$ open sets. The large time Dirichlet heat kernel estimates for unbounded open sets are different depending on the geometry of D as one sees for the cases of the symmetric α-stable processes and of the relativistic stable processes in [16] and in [10, 11], respectively.

Motivated by [16, 11], we establish the global sharp two-sided estimates on $p_D(t,x,y)$ in the exterior $C^{1,\eta}$ open set, that is, $C^{1,\eta}$ open set which is D^c is compact. It can be disconnected and in this case, there are bounded connected components. The number of the such bounded connected components is finite.

Theorem 1.2 *Let J be the symmetric function defined in* (1.2) *and Y be the symmetric pure jump Hunt process with the jumping intensity kernel J. Let $d > 2 \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}} + \alpha \cdot \mathbf{1}_{\{\beta=0\}}$, $T > 0$ and $R > 0$ be positive constants. For any $\eta \in (\alpha/2, 1]$, let D be an exterior $C^{1,\eta}$ open set in $\mathbb{R}^d$ with $C^{1,\eta}$ characteristics (r_0, Λ_0) and $D^c \subset B(0,R)$. Let D_0 be an unbounded connected component and $D_1, \dots, D_n$ be bounded connected components such that $D_0 \cup D_1 \cup \ldots \cup D_n = D$. Then for any $t \geq T$ ($t > 0$ when $\beta = 0$, respectively) and $x, y \in D$, the transition density $p_D(t,x,y)$ of Y^D has the following estimates.*

(1) *For any $\beta \in [0,\infty]$, there are positive constants $c_i = c_i(\alpha,\beta,\eta,r_0,\Lambda_0,R,T,d,L_3,L_4,\psi)$ ($c_i = c_i(\alpha,\beta,\eta,r_0,\Lambda_0,R,d,L_3,L_4,\psi)$ when $\beta = 0$, respectively), $i = 1,2$ such that*

$$p_D(t,x,y) \leq c_1 \left(1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \Psi^2_{c_2,T}(t, |x-y|).$$

(2) *Suppose that $\beta \in [0,1]$ or $\beta \in (1,\infty]$ with $|x-y| < 4/5$. Then there are positive constants $c_i = c_i(\alpha,\beta,\eta,r_0,\Lambda_0,R,T,d,L_3,L_4,\psi)$ ($c_i = c_i(\alpha,\beta,\eta,r_0,\Lambda_0,R,d,L_3,L_4,\psi)$ when $\beta = 0$, respectively), $i = 1,2$ such that*

$$p_D(t,x,y) \geq c_1 \left(1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \Psi^2_{c_2,T}(t, |x-y|).$$

(3) *Suppose that $\beta \in (1,\infty]$ with $|x-y| \geq 4/5$ and x, y are in a same component of D.*

(3.a) (Unbounded connected component) *There are positive constants $c_i = c_i(\alpha,\beta,\eta,r_0,\Lambda_0, R,T,d, L_3,L_4,\psi)$, $i = 1,2$ such that for $x, y \in D_0$*

$$p_D(t,x,y) \;\geq\; c_1 \left(1 \wedge \delta_D(x)\right)^{\alpha/2} \left(1 \wedge \delta_D(y)\right)^{\alpha/2} \Psi^2_{c_2,T}(t, |x-y|).$$

(3.b) (Bounded connected component) *There is a positive constant $c = c(\alpha,\beta,\eta,r_0,\Lambda_0, R,T,d,L_3,L_4,\psi)$ such that if $x, y \in D_j$ for some $j = 1, \dots n$,*

$$p_D(t,x,y) \;\geq\; c\, e^{-t\,\lambda_j}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2}$$

where $-\lambda_j < 0$ is the largest eigenvalue of the generator Y^{D_j}, $j = 1 \dots, n$.

(4) *Suppose that* $\beta \in (1,\infty)$ *with* $|x-y| \ge 4/5$ *and* x, y *are in different components of* D*. Then there are positive constants* $c_i = c_i(\alpha, \beta, \eta, r_0, \Lambda_0, R, T, d, L_3, L_4, \psi, \lambda_1, \dots, \lambda_n)$, $i = 1, 2$ *such that*

$$p_D(t,x,y) \ge c_1 \, (1 \wedge \delta_D(x))^{\alpha/2} \, (1 \wedge \delta_D(y))^{\alpha/2} \, \frac{\exp\left(-c_2(|x-y|^\beta + t)\right)}{|x-y|^{d+\alpha}}$$

where $-\lambda_j < 0$ *is the largest eigenvalue of the generator* Y^{D_j}, $j = 1 \dots, n$.

For a connected exterior $C^{1,\eta}$ open set, we can rewrite the sharp two-sided estimates on $p_D(t,x,y)$ for all $t > 0$ in a simple form combining Theorem 1.1(1)–(2) and Theorem 1.2(1)–(3a).

Corollary 1.3 *Let* J *be the symmetric function defined in* (1.2) *and* Y *be the symmetric pure jump Hunt process with the jumping intensity kernel* J*. Let* $d > 2 \cdot \mathbf{1}_{\{\beta \in (0,\infty]\}} + \alpha \cdot \mathbf{1}_{\{\beta = 0\}}$, $T > 0$ *and* $R > 0$ *be positive constants. For any* $\eta \in (\alpha/2, 1]$*, let* D *be a connected exterior* $C^{1,\eta}$ *open set in* $\mathbb{R}^d$ *with* $C^{1,\eta}$ *characteristics* (r_0, Λ_0) *and* $D^c \subset B(0,R)$*. Then there are positive constants* $c_i = c_i(\alpha, \beta, \eta, r_0, \Lambda_0, R, T, d, L_3, L_4, \psi) > 1, i = 1, 2$ *such that for every* $(t,x,y) \in (0,\infty) \times D \times D$*, we have*

$$p_D(t,x,y) \le c_1 \left(1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \cdot \begin{cases} \Psi^1_{c_2^{-1}, \gamma^{-1}, T}(t, |x-y|/6) & \text{if } t \in (0,T], \\ \Psi^2_{c_2^{-1}, T}(t, |x-y|) & \text{if } t \in [T, \infty), \end{cases}$$

and in addition D *is a connected, we have*

$$p_D(t,x,y) \ge c_1^{-1} \left(1 \wedge \frac{\delta_D(x)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{1 \wedge t^{1/\alpha}}\right)^{\alpha/2} \cdot \begin{cases} \Psi^1_{c_2, \gamma, T}(t, 5|x-y|/4) & \text{if } t \in (0,T], \\ \Psi^2_{c_2, T}(t, |x-y|) & \text{if } t \in [T, \infty) \end{cases}$$

where γ *is the constant in Theorem 1.1.*

By integrating the heat kernel estimates in Corollary 1.3 with respect to $t \in (0,\infty)$, one gets the following sharp two-sided Green function estimates of Y^D in the connected exterior $C^{1,\eta}$ open sets.

Corollary 1.4 *Let* J *be the symmetric function defined in* (1.2) *and* Y *be the symmetric pure jump Hunt process with the jumping intensity kernel* J*. Let* $d > 2 \cdot \mathbf{1}_{\{\beta \in (0,\infty]\}} + \alpha \cdot \mathbf{1}_{\{\beta = 0\}}$ *and* $R > 0$ *be a positive constant. For any* $\eta \in (\alpha/2, 1]$*, let* D *be a connected exterior* $C^{1,\eta}$ *open set in* $\mathbb{R}^d$ *with* $C^{1,\eta}$ *characteristics* (r_0, Λ_0) *and* $D^c \subset B(0,R)$*. Then there is a positive constant* $c = c(\alpha, \beta, \eta, r_0, \Lambda_0, R, d, L_3, L_4, \psi) > 1$ *such that for every* $(x,y) \in D \times D$*, we have*

$$\begin{aligned} & c^{-1} \left(\frac{1}{|x-y|^{d-\alpha}} + \frac{1}{|x-y|^{d-2}} \cdot \mathbf{1}_{\{\beta \in (0,\infty]\}}\right) \left(1 \wedge \frac{\delta_D(x)}{|x-y| \wedge 1}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y| \wedge 1}\right)^{\alpha/2} \\ \le {} & G_D(x,y) \le c \left(\frac{1}{|x-y|^{d-\alpha}} + \frac{1}{|x-y|^{d-2}} \cdot \mathbf{1}_{\{\beta \in (0,\infty]\}}\right) \left(1 \wedge \frac{\delta_D(x)}{|x-y| \wedge 1}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y| \wedge 1}\right)^{\alpha/2}. \end{aligned}$$

The approach developed in [11] provides us a main road map. By checking the cases depending on the value of β and the distance between x and y carefully, we establish sharp two-sided estimates on $p_D(t,x,y)$ for exterior $C^{1,\eta}$ open sets for all $t \in [T,\infty)$. In section 2, we first give elementary results on the functions $\Psi^1(t,r)$ and $\Psi^2(t,r)$ which are defined in (1.4) and (1.5). Also, we give the proof of the upper bound estimates on $p_D(t,x,y)$. In Section 3, we present the interior lower bound estimates on $p_{\overline{B}_R^c}(t,x,y)$ where $B_R := B(x_0,R)$ for some $x_0 \in \mathbb{R}^d$. In Section 4, the full lower bound estimates on $p_D(t,x,y)$ for exterior open set D are established by considering the cases whether the points are in a same component or in different components separately. The proof of Corollary 1.4 is given in Section 5.

Throughout this paper, the positive constants $C_1, C_2, L_1, L_2, L_3, L_4, \gamma_1, \gamma_2, \gamma$ will be fixed. In the statements of results and the proofs, the constants $c_i = c_i(a,b,c,\ldots), i = 1,2,3,\ldots,$ denote generic constants depending on $a,b,c,\ldots$ and there are given anew in each statement and each proof. The dependence of the constants on the dimension d, on $\alpha \in (0,2)$ and on the positive constants $L_1, L_2, L_3, L_4, \gamma_1, \gamma_2, \gamma$ will not be mentioned explicitly.

2 Upper bound estimates

We first give elementary lemmas which are used several times to estimates the upper and lower bound on $p_D(t,x,y)$ where $t \geq T$ ($t > 0$ when $\beta = 0$, respectively). Recall the functions $\Psi^1(t,r)$ and $\Psi^2(t,r)$ which are defined in (1.4) and (1.5).

Lemma 2.1 *Let $t_0 > 0$ and $a,b,c \geq 1$ be fixed constants. For any $\beta \in (0,\infty]$, suppose that N_1, N_2 be positive constants satisfying $N_2 \geq N_1 \cdot (ab \vee c^{2/\beta})$. Then there exist positive constants $c_i = c_i(t_0), i = 1,2$ such that for every $r > 0$, we have that*

$$(1) \qquad \Psi^1_{b^{-1},c^{-1},t_0}(t_0, N_1^{-1}r) \leq c_1 \Psi^1_{a,c,t_0}(t_0, N_2^{-1}r) \qquad \text{and}$$
$$(2) \qquad \Psi^1_{a^{-1},c^{-1},t_0}(t_0, N_2 r) \leq c_2 \Psi^1_{b,c,t_0}(t_0, N_1 r).$$

Proof. When $\beta \in (0,1]$, since $N_2 \geq N_1 c^{2/\beta}$, we have (1) and (2).

When $\beta \in (1,\infty]$, since $t_0^{-d/\alpha} \wedge t_0 r^{-d/\alpha} \asymp 1$ for any $r < 1$, we only consider the case $1 \leq N_2^{-1}r (\leq N_1^{-1}r)$ to prove (1) and $1 \leq N_1 r (\leq N_2 r)$ to prove (2). In these cases, since $\log x$ is increasing in x and $N_2 \geq N_1 ab$, we have (1) and (2). $\Box$

Lemma 2.2 *Let T, a and b be positive constants. (1) If $b \geq 1$, there exists a positive constant $c = c(b)$ such that for every $t \in [T,\infty)$ and $r > 0$, we have that*

$$\Psi^2_{a,T}(t, b^{-1}r) \leq \Psi^2_{ab^{-2},T}(t,r).$$

(2) In addition, for $a,b \geq 1$ and $\beta \in (0,\infty]$, suppose that N be a positive constant satisfying $N \geq (ab)^{1/(\beta\wedge 1)}$. Then for every $t \in [T,\infty)$ and $r > 0$, we have that

$$\Psi^2_{b^{-1},T}(t,r) \leq \Psi^2_{a,T}(t, N^{-1}r).$$

Proof. Since $b \geq 1$, it is easy to prove (1) when $\beta \in [0,1]$. Also, since

$$b\left(1+\log^+\frac{Tb^{-1}r}{t}\right) \geq (1+\log b)\cdot\left(1+\log^+\frac{Tb^{-1}r}{t}\right) \geq \left(1+\log^+\frac{Tr}{t}\right),$$

for any $b \geq 1$, we have (1) when $\beta \in (1,\infty]$.

On the other hand, since $N \geq (ab)^{1/\beta}(\geq 1)$, we have that

$$b^{-1}\left(r^\beta \wedge \frac{r^2}{t}\right) \geq b^{-1}N^\beta\left((N^{-1}r)^\beta \wedge \frac{(N^{-1}r)^2}{t}\right) \geq a\left((N^{-1}r)^\beta \wedge \frac{(N^{-1}r)^2}{t}\right). \tag{2.1}$$

Also, since $r \to 1+\log^+ r$ is non-decreasing and $N \geq ab\,(\geq 1)$, we have that

$$\begin{aligned} b^{-1}\left(r\left(1+\log^+\frac{Tr}{t}\right)^{(\beta-1)/\beta} \wedge \frac{r^2}{t}\right) &\geq b^{-1}N\left(N^{-1}r\left(1+\log^+\frac{N^{-1}Tr}{t}\right)^{(\beta-1)/\beta} \wedge \frac{(N^{-1}r)^2}{t}\right)\\ &\geq a\left(N^{-1}r\left(1+\log^+\frac{N^{-1}Tr}{t}\right)^{(\beta-1)/\beta} \wedge \frac{(N^{-1}r)^2}{t}\right). \end{aligned} \tag{2.2}$$

Hence, by (2.1) for $\beta \in (0,1]$ and by (2.2) for $\beta \in (1,\infty]$, we have (2). □

We now prove the upper bound estimates in Theorem 1.2(1).

Proof of Theorem 1.2(1) When $\beta = 0$, by Theorem 1.1(1), we may assume that $t \geq T$. Wthout loss of the generality, we may assume that $T = 3$. By the semigroup property and Theorem 1.1(1), we have that for $t-2 \geq 1$ and $x, y \in D$,

$$\begin{aligned} p_D(t,x,y) &= \int_D\int_D p_D(1,x,z)p_D(t-2,z,w)p_D(1,w,y)dzdw\\ &\leq c_1\,(1\wedge\delta_D(x))^{\alpha/2}\,(1\wedge\delta_D(y))^{\alpha/2}\,f_1(t,x,y) \end{aligned} \tag{2.3}$$

where C_2 and γ are given constants in Theorem 1.1 and

$$f_1(t,x,y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \Psi^1_{C_2^{-1},\gamma^{-1},1}(1,|x-z|/6)\,p(t-2,z,w)\,\Psi^1_{C_2^{-1},\gamma^{-1},1}(1,|y-w|/6)dzdw. \tag{2.4}$$

Let $A_1 := \max\{C_1^{2/(\beta\wedge1)}, 6\gamma^{2/\beta}, 6C_1C_2\}$ ($A_1 = 6$ when $\beta = 0$, respectively) where C_1 is given constant in (1.6) and (1.7). Then by (1.7), there exists constants $c_i = c_i(\beta) > 0$, $i = 2,3$ such that

$$p(t-2,z,w) \leq c_2\,\Psi^2_{C_1^{-1},1}(t-2,|z-w|) \leq c_2\,\Psi^2_{C_1,1}(t-2,A_1^{-1}|z-w|) \leq c_3\,p(t-2,A_1^{-1}z,A_1^{-1}w).$$

For the second inequaltiy, when $\beta \in (0,\infty]$, we use (2) in Lemma 2.2 with $N = A_1$, $a = b = C_1$ and the fact $A_1 \geq C_1^{2/(\beta\wedge1)}$. When $\beta = 0$, the second inequality holds since $A_1 \geq 1$.

Also, by (1.6), there exist constants $c_i = c_i(\beta) > 0, i = 4,5$ such that

$$\begin{aligned} &\Psi^1_{C_2^{-1},\gamma^{-1},1}(1,|x-z|/6) \leq c_4\,\Psi^1_{C_1\gamma,1}(1,A_1^{-1}|x-z|) \leq c_5\,p(1,A_1^{-1}x,A_1^{-1}z) \quad\text{and}\\ &\Psi^1_{C_2^{-1},\gamma^{-1},1}(1,|y-w|/6) \leq c_4\,\Psi^1_{C_1\gamma,1}(1,A_1^{-1}|y-w|) \leq c_5\,p(1,A_1^{-1}y,A_1^{-1}w). \end{aligned}$$

For the first inequalties above, when $\beta \in (0, \infty]$, we use (1) in Lemma 2.1 along with $a = C_1$, $b = C_2$, $c = \gamma$, $N_1 = 6$ and $N_2 = A_1$ and the fact $A_1 \geq 6(C_1 C_2 \vee \gamma^{2/\beta})$. When $\beta = 0$, the first inequalities hold since $A_1 = 6$.

Applying the above observations to (2.4) and by the change of variable $\hat{z} = A_1^{-1} z$, $\hat{w} = A_1^{-1} w$, the semigroup property and (1.7), we conclude that

$$
\begin{aligned}
f_1(t,x,y) &\leq c_6 \int_{\mathbb{R}^d \times \mathbb{R}^d} p(1, A_1^{-1}x, \hat{z})\, p(t-2, \hat{z}, \hat{w}) p(1, A_1^{-1}y, \hat{w}) d\hat{z} d\hat{w} \\
&= c_6\, p(t, A_1^{-1}x, A_1^{-1}y) \leq c_7 \Psi^2_{C_1^{-1}, T}(t, A_1^{-1}|x-y|) \\
&\leq c_8 \Psi^2_{C_1^{-1}A_1^{-2}, T}(t, |x-y|). \qquad (2.5)
\end{aligned}
$$

We have applied (1) in Lemma 2.2 with $a = C_1$ and $b = A_1$ for the last inequality. Applying (2.5) to (2.3), we have proved the upper bound estimates in Theorem 1.2. □

3 Interior lower bound estimates

The goal of this section is to the establish interior lower bound estimate on the heat kernel $p_{\overline{B}_R^c}(t,x,y)$ for $t \geq T$ ($t > 0$ when $\beta = 0$, respectively) where $B_R = B(x_0, R)$ for some $R > 0$ and $x_0 \in \mathbb{R}^d$. We will combine ideas from [10] and [21].

First, we introduce a Lemma which will be used in the proof of Lemma 3.2 and Proposition 3.3. Let $\varphi(r) := r^2 \cdot \mathbf{1}_{\{\beta \in (0,\infty]\}} + r^\alpha \cdot \mathbf{1}_{\{\beta = 0\}}$ and then $\varphi^{-1}(t) = t^{1/2} \cdot \mathbf{1}_{\{\beta \in (0,\infty]\}} + t^{1/\alpha} \cdot \mathbf{1}_{\{\beta = 0\}}$.

Lemma 3.1 *Let a be a positive constant and $T > 0$ and $\beta \in [0, \infty]$. Then there exists a positive constant $c = c(a, \beta, T)$ ($c = c(a)$ when $\beta = 0$, respectively) such that for all $t \in [T, \infty)$ ($t > 0$ when $\beta = 0$, respectively), we have*

$$
\inf_{y \in \mathbb{R}^d} \mathbb{P}_y \left(\tau_{B(y, a\varphi^{-1}(t))} > t \right) \geq c.
$$

Proof. When $\beta = 0$, using [15, Theorem 4.12 and Proposition 4.9], the proof is almost identical to that of [9, Lemma 3.1]. When $\beta \in (0, \infty]$, using [6, Theorem 4.8], the proof is the same as that of [10, Lemma 3.2]. So we omit the proof detail. □

Lemma 3.2 *Let D be an arbitrary open set. Suppose that a be a positive constant and $T > 0$ and $\beta \in [0, \infty]$. Then there exists a positive constant $c = c(a, \beta, T)$ ($c = c(a)$ when $\beta = 0$, respectively) such that for all $t \in [T, \infty)$ ($t > 0$ when $\beta = 0$, respectively) and $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geq a\varphi^{-1}(t)$ and $|x - y| \geq 2^{-1} a \varphi^{-1}(t)$, we have*

$$
\mathbb{P}_x \left(Y_t^D \in B\big(y,\, 2^{-1} a \varphi^{-1}(t)\big) \right) \geq c\, t \cdot \varphi^{-d}(t) j(|x-y|).
$$

Proof. Using Lemma 3.1, the strong Markov property and Lévy system (1.3), the proof of the lemma is similar to that of [21, Proposition 3.3]. So we omit the proof detail. □

For the remainder of this section, we assume that D is a domain with the following property: there exist $\lambda_1 \in [1, \infty)$ and $\lambda_2 \in (0, 1]$ such that for every $r \le 1$ and x, y in the same component of D with $\delta_D(x) \wedge \delta_D(y) \ge r$, there exists in D a length parameterized rectifiable curve l connecting x to y with the length $|l|$ of l is less than or equal to $\lambda_1|x-y|$ and $\delta_D(l(u)) \ge \lambda_2 r$ for $u \in (0, |l|]$. Clearly, such a property holds for all $C^{1,\eta}$ domains with compact complements, and domains above graphs of $C^{1,\eta}$ functions.

The following Propositions are motivated by [10].

Proposition 3.3 *Let a be a positive constant and $T > 0$ and $\beta \in [0, \infty]$. Then there exists a positive constant $c = c(a, \beta, T, \lambda_1, \lambda_2)$ ($c = c(a, \lambda_1, \lambda_2)$ when $\beta = 0$, respectively) such that for all $t \in [T, \infty)$ ($t > 0$ when $\beta = 0$, respectively) and $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \ge a\varphi^{-1}(t) \ge 2|x-y|$, we have $p_D(t, x, y) \ \ge \ c/\varphi^{-d}(t)$.*

Proof. By the same proof as that of [10, Proposition 3.4], we deduce the proposition using the parabolic Harnack inequality(see [15, Theorem 4.12] for $\beta = 0$ and [6, Theorem 4.11] for $\beta \in (0, \infty]$) and Lemma 3.2. □

Proposition 3.4 *Let a be a positive constant and $T > 0$ and $\beta \in [0, \infty]$. Then there exists a positive constant $c = c(a, \beta, T, \lambda_1, \lambda_2)$ ($c = c(a, \lambda_1, \lambda_2)$ when $\beta = 0$, respectively) such that for all $t \in [T, \infty)$ ($t > 0$ when $\beta = 0$, respectively) and $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \ge a\varphi^{-1}(t)$ and $|x-y| \ge 2^{-1}a\varphi^{-1}(t)$, we have $p_D(t, x, y) \ge c\, t j(|x-y|)$.*

Proof. By the same proof as that of [10, Proposition 3.5], we deduce the proposition using the semigroup property, Lemma 3.2 and Proposition 3.3. □

Also, since the proof of the following proposition is almost identical to that of [10, Proposition 3.6] using Proposition 3.3, we skip the proof.

Proposition 3.5 *Let $\beta \in (1, \infty]$ and a and C_* be positive constants. Then there exist positive constants $c_i = c_i(a, \beta, C_*, \lambda_1, \lambda_2)$, $i = 1, 2$ such that for every $t \in (0, \infty)$ and $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \ge a\sqrt{t}$, we have*

$$p_D(t, x, y) \ \ge \ c_1\, t^{-d/2} \exp\left(-c_2 \frac{|x-y|^2}{t}\right) \ \text{ when } \ C_*|x-y| \le t \le |x-y|^2.$$

Now, we estimates the interior lower bound for $p_D(t, x, y)$ where $\beta \in (1, \infty]$ and $T \le t \le C_*T|x-y|$ for any positive constant $C_* < 1$. The following Proposition 3.6 and Proposition 3.7 are counterparts of [21, Propsition 3.6] and [21, Proposition 3.5], respectively. (See, also [6, Theorem 5.5]) and [4, Theorem 3.6], respectively.)

Proposition 3.6 *Let $\beta \in (1,\infty)$ and a, T and $C_* \in (0,1)$ be positive constants. Then there exist positive constants $c_i = c_i(a,\beta,T,C_*,\lambda_1,\lambda_2)$, $i = 1,2$ such that for every $t \in [T,\infty)$ and $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geq a\sqrt{t}$, we have*

$$p_D(t,x,y) \ \geq\ c_1 \exp\left(-c_2|x-y|\left(1+\log\frac{T|x-y|}{t}\right)^{(\beta-1)/\beta}\right) \ \textit{when}\ \ C_*T|x-y| \geq t.$$

Proof. We let $r := |x-y|$ and fix $C_* \in (0,1)$. Note that $r \geq C_*^{-1}t/T > t/T \geq 1$ and $r\exp(-r^\beta) \leq \exp(-1)(<1)$ for $\beta > 1$. So we only consider the case $Tr\exp(-r^\beta) < t$ $(\leq C_*Tr)$ which is equivalent to $r\left(\log(Tr/t)\right)^{-1/\beta} > 1$. Let $k \geq 2$ be a positive integer such that

$$1 < r\left(\log\frac{Tr}{t}\right)^{-1/\beta} \leq k < r\left(\log\frac{Tr}{t}\right)^{-1/\beta} + 1 < 2r\left(\log\frac{Tr}{t}\right)^{-1/\beta}. \tag{3.1}$$

Then we have that

$$\frac{t}{k} \leq \frac{t}{r}\left(\log\frac{Tr}{t}\right)^{1/\beta} \leq T\cdot\sup_{s\geq C_*^{-1}} s^{-1}(\log s)^{1/\beta} =: t_0 < \infty \tag{3.2}$$

By our assumption on D, there is a length parameterized curve $l \subset D$ connecting x and y such that the total length $|l|$ of l is less than or equal to $\lambda_1 r$ and $\delta_D(l(u)) \geq \lambda_2 a\sqrt{t}$ for every $u \in [0,|l|]$. We define $r_t := (2^{-1}\lambda_2 a\sqrt{t}) \wedge ((6\lambda_1)^{-1}(\log(Tr/t))^{1/\beta})$. Then by (3.1) and the assumption $\log(C_*^{-1}) < \log(Tr/t)$, we have that

$$0 < r_0 := \left(\frac{\lambda_2 a\sqrt{T}}{2}\right) \wedge \left(\frac{(\log C_*^{-1})^{1/\beta}}{6\lambda_1}\right) \leq r_t \leq \frac{1}{6\lambda_1}\left(\log\frac{Tr}{t}\right)^{1/\beta} < \frac{r}{3\lambda_1 k}. \tag{3.3}$$

Define $x_i := l(i|l|/k)$ and $B_i := B(x_i, r_t)$ for $i = 0,1,2,\ldots,k$ then $\delta_D(x_i) \geq \lambda_2 a\sqrt{t} > r_t$ and $B_i \subset D$. For every $y_i \in B_i$, we have that $\delta_D(y_i) \geq 2^{-1}\lambda_2 a\sqrt{t} > 2^{-1}\lambda_2 a\sqrt{t/k}$ and

$$|y_i - y_{i+1}| \leq |x_i - x_{i+1}| + 2r_t \leq \left(\lambda_1 + \frac{2}{3\lambda_1}\right)\frac{r}{k}. \tag{3.4}$$

Thus by Proposition 3.3 and 3.4 along with the definition of j, (3.1), (3.2) and (3.4), there exist constants $c_i > 0, i = 1,\ldots,5$ such that

$$\begin{aligned} p_D(t/k, y_i, y_{i+1}) &\geq c_1\left(\left(\frac{t}{k}\right)^{-d/2} \wedge \frac{t}{k}\cdot j(|y_i - y_{i+1}|)\right) \ \geq\ c_2\left(1 \wedge \Big(\frac{t}{k}\frac{e^{-c_3(r/k)^\beta}}{(r/k)^{d+\alpha}}\Big)\right) \\ &\geq\ c_4\frac{t}{Tr}\left(\frac{k}{r}\right)^{d+\alpha-1} e^{-c_3(r/k)^\beta} \ \geq\ c_4\frac{t}{Tr}\left(\log\frac{Tr}{t}\right)^{-\frac{d+\alpha-1}{\beta}}\left(\frac{t}{Tr}\right)^{c_3} \ \geq\ c_4\left(\frac{t}{Tr}\right)^{c_5}. \end{aligned} \tag{3.5}$$

Therefore, by the semigroup property, (3.3) and (3.5), we conclude that

$$p_D(t,x,y) \geq \int_{B_1}\cdots\int_{B_{k-1}} p_D(t/k,x,y_1)\cdots p_D(t/k,y_{k-1},y)dy_1\cdots dy_{k-1}$$

$$
\begin{aligned}
&\geq \left(c_4\left(\frac{t}{Tr}\right)^{c_5}\right)^k \Pi_{i=1}^{k-1}|B_i| \;\geq\; \left(\frac{c_6 t}{Tr}\right)^{c_5 k} \\
&\geq c_7 \exp\left(-c_5 k\left(\log\frac{Tr}{c_8 t}\right)\right) \geq c_7 \exp\left(-c_9 r\left(\log\frac{Tr}{t}\right)^{1-1/\beta}\right) \\
&\geq c_7 \exp\left(-c_9 r\left(1+\log\frac{Tr}{t}\right)^{1-1/\beta}\right).
\end{aligned}
$$

□

Proposition 3.7 *Let $\beta=\infty$ and a, T and $C_* \in (1/2,1)$ be positive constants. Then there exist positive constants $c_i = c_i(a,T,C_*,\lambda_1,\lambda_2)$, $i=1,2$ such that for every $t\in[T,\infty)$ and $x,y\in D$ with $\delta_D(x)\wedge\delta_D(y)\geq a\sqrt{t}$, we have*

$$
p_D(t,x,y) \;\geq\; c_1 \exp\left(-c_2|x-y|\left(1+\log\frac{T|x-y|}{t}\right)\right) \;\; \text{when} \;\; C_*T|x-y|\geq t.
$$

Proof. Let $r:=|x-y|$ and fix $C_*\in(1/2,1)$. Since $T\leq t\leq C_*Tr$, we note that $1\leq C_*r$. By our assumption on D, there is a length parameterized curve $l\subset D$ connecting x and y such that the total length $|l|$ of l is less than or equal to $\lambda_1 r$ and $\delta_D(l(u))\geq \lambda_2 a\sqrt{t}$ for every $u\in[0,|l|]$. Let $k\geq 2$ be a positive integer satisfying

$$
1 < 8\lambda_1 C_* r \leq k < 8\lambda_1 C_* r + 1 \leq (8\lambda_1+1)C_* r. \tag{3.6}
$$

Define $r_t := (\lambda_2 a\sqrt{t}/2)\wedge 8^{-1}$, $x_i := l(i|l|/k)$ and $B_i := B(x_i, r_t)$ for $i=0,1,\dots,k$. Then $\delta_D(x_i) > 2r_t$ and $B_i\subset B(x_i,2r_t)\subset D$. For every $y_i\in B_i$, since $t/k < t/(8\lambda_1 C_* r)\leq T/(8\lambda_1)$, we have $\delta_D(y_i) > r_t > c_1\sqrt{t/k}$ for some constant $c_1 = c_1(a,T,\lambda_1,\lambda_2)>0$. Also, for each $y_i\in B_i$,

$$
|y_i-y_{i+1}| \leq |y_i-x_i| + |x_i-x_{i+1}| + |x_{i+1}-y_{i+1}| \leq \frac{1}{8}+\frac{|l|}{k}+\frac{1}{8} < \frac{\lambda_1 r}{8\lambda_1 C_* r}+\frac{1}{4}\leq\frac{1}{2}. \tag{3.7}
$$

By Proposition 3.3 and 3.4 along with the definition of j, (3.7) and the fact that $t/k < T/(8\lambda_1)$, there are constants $c_i = c_i(a,T,\lambda_1)>0$, $i=2,\dots,4$, such that for $(y_i,y_{i+1})\in B_i\times B_{i+1}$,

$$
p_D(t/k,y_i,y_{i+1}) \geq c_2\left((t/k)^{-d/\alpha}\wedge\frac{t/k}{|y_i-y_{i+1}|^{d+\alpha}}\right) \geq c_3\,(1\wedge t/k) \geq c_4\, t/(Tk). \tag{3.8}
$$

Thus, by the semigroup property combining the fact $r_t\geq r_T\wedge 8^{-1}$, (3.6) and (3.8), we obtain that

$$
\begin{aligned}
p_D(t,x,y) &\geq \int_{B_1}\dots\int_{B_{k-1}} p_D(t/k,x,y_1)\dots p_D(t/k,y_{k-1},y)dy_{k-1}\dots dy_1 \geq \left(\frac{c_4\, t}{Tk}\right)^k \Pi_{i=1}^{k-1}|B_i| \\
&\geq \left(\frac{c_5\, t}{Tk}\right)^k \geq c_6\left(\frac{c_7\, t}{Tr}\right)^k \geq c_6 \exp\left(-c_8 r\log\frac{Tr}{c_7 t}\right) \geq \exp\left(-c_9 r\left(1+\log\frac{Tr}{t}\right)\right).
\end{aligned}
$$

□

Recall that $B_R = B(x_0,R)$. Note that a exterior ball $\overline{B}_R^c$ is a domain in which the path distance is comparable to the Euclidean distance with characteristics (λ_1,λ_2) independent of x_0 and R. Hence, the previous propositions yield the following Theorem.

Theorem 3.8 *Let a and T be positive constants. Then for any $\beta \in [0,\infty]$, there exists positive constants $c_i = c_i(a,\beta,T)$ ($c = c(a)$ when $\beta = 0$, respectively) , $i = 1,2$, such that for every $R > 0$, $t \in [T,\infty)$ ($t > 0$ when $\beta = 0$, respectively) and $x, y \in \overline{B}_R^c$ with $\delta_{\overline{B}_R^c}(x) \wedge \delta_{\overline{B}_R^c}(y) \geq a\varphi^{-1}(t)$, we have*

$$p_{\overline{B}_R^c}(t,x,y) \;\geq\; c_1 \Psi^2_{c_2,T}(t,|x-y|)$$

where $\Psi^2_{c_2,T}(t,r)$ is defined in (1.4).

Proof. Let $r := |x-y|$. For any $\beta \geq 0$, if $\varphi(r) < t$, by Proposition 3.3, we have the conclusion.

Suppose $t \leq \varphi(r)$. When $\beta \in [0,1]$, we have the conclusion by Proposition 3.4 and Proposition 3.5. When $\beta \in (1,\infty)$, using Proposition 3.5 and Proposition 3.6 , and when $\beta = \infty$, using Proposition 3.5 and Proposition 3.7, we have the conclusion. □

4 Lower bound estimates

In this section, we assume that the dimension $d > 2 \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}} + \alpha \cdot \mathbf{1}_{\{\beta=0\}}$. To establish the lower bound estimates in Theorem 1.2(2)–(4), we first consider the lower bound estimates on $p_{\overline{B}_R^c}(t,x,y)$ for $t \geq T$ ($t > 0$ when $\beta = 0$, respectively) where B_R is a ball of radius $R > 0$ centered at x_0. Since all following estimates are independent of x_0, we may assume that $x_0 = 0$.

We define the Green function $G(x,y)$ of Y in $\mathbb{R}^d$ as $G(x,y) := \int_0^\infty p(t,x,y)dt$ for every $x, y \in \mathbb{R}^d$. Then by the fact that $\int_0^\infty (t^{-d/\alpha} \wedge tr^{-d-\alpha})dt \asymp r^{\alpha-d}$ for $d > \alpha$ when $\beta = 0$ and by [6, Theorem 6.1] when $\beta \in (0,\infty]$, we have that

$$G(x,y) \asymp \left(|x-y|^{\alpha-d} + |x-y|^{2-d} \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}}\right). \tag{4.1}$$

For any Borel set $A \subset \mathbb{R}^d$, define the first exit time of A as $\tau_A = \inf\{t > 0 : Y_t \notin A\}$ and the first hitting time of A as $T_A = \inf\{t > 0 : Y_t \in A\}$. The next lemma provide us the beginning point for the lower bound estimates which proof is almost identical to that of [11, Lemma 4.1] using (4.1), so we omit the proof.

Lemma 4.1 *There is a constant $C_3 > 1$ such that for all $R > 0$,*

$$C_3^{-1}\frac{R^d}{R^\alpha + R^2}\left(|x|^{\alpha-d} + |x|^{2-d} \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}}\right) \leq \mathbb{P}_x(T_{\overline{B}_R} < \infty)$$
$$\leq C_3\frac{R^d}{R^\alpha + R^2}\left(|x|^{\alpha-d} + |x|^{2-d} \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}}\right), \qquad \text{for } \; |x| \geq 2R.$$

The following ideas of obtaining the lower bound estimates on $p_{\overline{B}_R^c}(t,x,y)$ are motivated by that of Section 5 in [11] and for the sake of completeness, we give proofs detail. For the simplicity of the notation, hereafter for any $y \in \mathbb{R}^d \backslash \{0\}$ and $r > 0$, we define $H(y,r) := \{z \in B(y,r) : z \cdot y \geq 0\}$. Recall that $\varphi(r) = r^2 \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}} + r^\alpha \cdot \mathbf{1}_{\{\beta=0\}}$ and $\varphi^{-1}(t) = t^{1/2} \cdot \mathbf{1}_{\{\beta\in(0,\infty]\}} + t^{1/\alpha} \cdot \mathbf{1}_{\{\beta=0\}}$.

Lemma 4.2 *Let T be a positive constant. Then for any $\beta \in [0,\infty]$, there exists constants $\varepsilon = \varepsilon(\beta,T) > 0$ and $M_1 = M_1(\beta,T) \geq 3$ ($\varepsilon > 0$ and $M_1 \geq 3$ when $\beta = 0$, respectively) such that the following holds: for any $R > 0$, $t \in [T,\infty)$ ($t > 0$ when $\beta = 0$, respectively) and x, y satisfying $|x| > M_1 R$, $|y| > R$ and $y \in B(x, 9\varphi^{-1}(t))$, we have*

$$\mathbb{P}_x\left(Y_t^{\overline{B}_R^c} \in H(y, \varphi^{-1}(t)/2)\right) \geq \varepsilon.$$

Proof. Applying (1.7) (Applying (1.6) and (1.7) when $\beta = 0$, respectively) and by the change of variable with $v = z/\varphi^{-1}(t)$, for any $t \geq T$ ($t > 0$ when $\beta = 0$, respectively), there are constants $c_i = c_i(\beta,T) > 0$ ($c_i > 0$ when $\beta = 0$, respectively), $i = 1, \cdots, 3$ such that

$$\begin{aligned}
&\mathbb{P}_x\left(Y_t \in H(y,\varphi^{-1}(t)/2)\right) \geq \inf_{w\in B(y,9\varphi^{-1}(t))} \mathbb{P}_w\left(Y_t \in H(y,\varphi^{-1}(t)/2)\right)\\
&\quad\geq c_1 \inf_{w\in B(y,9\varphi^{-1}(t))} \int_{H(y,\varphi^{-1}(t)/2)} \Psi^2_{C_1,T}(t,|w-z|)\cdot \mathbf{1}_{\{\beta\in(0,\infty]\}} + \left(t^{-d/\alpha}\wedge t|w-z|^{-d-\alpha}\right)\cdot \mathbf{1}_{\{\beta=0\}}\,dz\\
&\quad\geq c_2 \inf_{w\in B(y,9\varphi^{-1}(t))} \int_{H(y,\varphi^{-1}(t)/2)} \frac{1}{\varphi^{-d}(t)}\left(\exp\left(-C_1\frac{|w-z|^2}{t}\right)\cdot \mathbf{1}_{\{\beta\in(0,\infty]\}} + \mathbf{1}_{\{\beta=0\}}\right) dz\\
&\quad= c_3 \inf_{w_0\in B(y_0,9)} \int_{H(y_0,1/2)} \exp\left(-C_1|w_0-v|^2\right)\cdot \mathbf{1}_{\{\beta\in(0,\infty]\}} + \mathbf{1}_{\{\beta=0\}}\,dv\\
&\quad\geq 2^{-1}c_3|B(0,1/2)|\left(e^{-C_1 10^2}\cdot \mathbf{1}_{\{\beta\in(0,\infty]\}} + \mathbf{1}_{\{\beta=0\}}\right)
\end{aligned}$$

where $y_0 := y/\varphi^{-1}(t)$ and $w_0 := w/\varphi^{-1}(t)$. When $\beta = 0$, since $|w-z| \leq 10t^{1/\alpha}$, the third inequality holds. Hence, there is $\varepsilon \in (0,1/4)$ so that for any $t \geq T$ ($t > 0$ when $\beta = 0$, respectively), $x \in \mathbb{R}^d$ and $y \in B(x, 9\varphi^{-1}(t))$, we have

$$\varepsilon < \frac{1}{2}\mathbb{P}_x\left(Y_t \in H(y,\varphi^{-1}(t)/2)\right). \tag{4.2}$$

For $d > 2\cdot\mathbf{1}_{\{\beta\in(0,\infty]\}} + \alpha\cdot\mathbf{1}_{\{\beta=0\}}$ and the constant $C_3 > 1$ in Lemma 4.1, we may choose $M_1 \geq 3$ so that $C_3(M_1^{2-d} + M_1^{\alpha-d}\cdot\mathbf{1}_{\{\beta\in(0,\infty]\}}) \leq \varepsilon$. For any x with $|x| > M_1 R$, by Lemma 4.1, we have that

$$\begin{aligned}
\mathbb{P}_x\left(\tau_{\overline{B}_R^c} \leq t\right) = \mathbb{P}_x\left(T_{\overline{B}_R} < \infty\right) &\leq C_3\frac{R^d}{R^\alpha + R^2}(|x|^{2-d} + |x|^{\alpha-d}\cdot\mathbf{1}_{\{\beta\in(0,\infty]\}})\\
&\leq C_3\left(\frac{R^2}{R^\alpha+R^2}M_1^{2-d} + \frac{R^\alpha}{R^\alpha+R^2}M_1^{\alpha-d}\cdot\mathbf{1}_{\{\beta\in(0,\infty]\}}\right)\\
&\leq C_3(M_1^{2-d} + M_1^{\alpha-d}\cdot\mathbf{1}_{\{\beta\in(0,\infty]\}}) \leq \varepsilon. \qquad (4.3)
\end{aligned}$$

Hence, combining (4.2) and (4.3), we obtain that

$$\begin{aligned}
\mathbb{P}_x\left(Y_t^{\overline{B}_R^c} \in H(y,\varphi^{-1}(t)/2)\right) &= \mathbb{P}_x\left(\tau_{\overline{B}_R^c} > t\right) - \mathbb{P}_x\left(Y_t^{\overline{B}_R^c} \notin H(y,\varphi^{-1}(t)/2); \tau_{\overline{B}_R^c} > t\right)\\
&\geq \mathbb{P}_x\left(\tau_{\overline{B}_R^c} > t\right) - \mathbb{P}_x\left(Y_t \notin H(y,\varphi^{-1}(t)/2)\right)\\
&\geq (1-\varepsilon) - (1-2\varepsilon) = \varepsilon.
\end{aligned}$$

$\square$

Lemma 4.3 *Let $T>0$, $\beta\in[0,\infty]$, and $M_1=M_1(\beta,T/8)\ge 3$ ($M_1\ge 3$ when $\beta=0$, respectively) be the constant in Lemma 4.2. Then there exists a positive constant $c=c(\beta,T)>0$ ($c>0$ when $\beta=0$, respectively) such that for any $R>0$, $t\in[T,\infty)$ ($t>0$ when $\beta=0$, respectively) and x,y satisfying $|x|>M_1R$, $|y|>M_1R$ and $|x-y|\le\varphi^{-1}(t)/6$, we have that $p_{\overline{B}_R^c}(t,x,y)\ge c/\varphi^{-d}(t)$.*

Proof. Without loss of generality we may assume that $|y|\ge|x|$. If $\delta_{\overline{B}_R^c}(y)>\varphi^{-1}(t)/2$, then $\delta_{\overline{B}_R^c}(x)\ge\delta_{\overline{B}_R^c}(y)-|x-y|\ge\varphi^{-1}(t)/3$, and hence the lemma follows immediately from Proposition 3.3.

Now we assume that $\delta_{\overline{B}_R^c}(y)\le\varphi^{-1}(t)/2$. By the semigroup property and the parabolic Harnack inequality(see [6, Theorem 4.11]) , we have

$$\begin{aligned} p_{\overline{B}_R^c}(t,x,y)\ge&\int_{H(y,\varphi^{-1}(t/2))}p_{\overline{B}_R^c}(t/2,x,z)p_{\overline{B}_R^c}(t/2,z,y)dz\\ &\ge c_1\mathbb{P}_x\left(Y_{t/2}^{\overline{B}_R^c}\in H(y,\varphi^{-1}(t/2))\right)p_{\overline{B}_R^c}\left(t/2-\varphi(2\delta_{\overline{B}_R^c}(y))/4,y,y\right).\end{aligned}\tag{4.4}$$

Note that $t\ge s:=t/2-\varphi(2\delta_{\overline{B}_R^c}(y))/4\ge t/4\ge T/4$ ($s\ge t/4>0$ when $\beta=0$, respectively). So by the semigroup property, the Cauchy-Schwarz inequality and Lemma 4.2, we obtain that

$$\begin{aligned} p_{\overline{B}_R^c}(s,y,y)\ge&\int_{H(y,\varphi^{-1}(s)/2)}\left(p_{\overline{B}_R^c}(s/2,y,z)\right)^2dz\\ \ge&\frac{2}{|B(y,\varphi^{-1}(s)/2)|}\mathbb{P}_y\left(Y_{s/2}^{\overline{B}_R^c}\in H(y,\varphi^{-1}(s)/2)\right)^2\ge c_2/\varphi^{-d}(s)\ge c_2/\varphi^{-d}(t).\end{aligned}\tag{4.5}$$

Applying Lemma 4.2 again and (4.5) to (4.4), we have that $p_{\overline{B}_R^c}(t,x,y)\ge c_3/\varphi^{-d}(t)$. □

Proposition 4.4 *Let $T>0$, $\beta\in[0,\infty]$, and $M_1=M_1(\beta,T/16)\ge 3$ ($M_1\ge 3$ when $\beta=0$, respectively) be the constant in Lemma 4.2. Then there exist positive constants $c=c(\beta,T)$ and $C_4=C_4(\beta,T)$ (c, $C_4>0$ when $\beta=0$, respectively) such that for any $R>0$, $t\in[T,\infty)$ ($t>0$ when $\beta=0$, respectively) and x,y satisfying $|x|>M_1R$, $|y|>M_1R$, we have that $p_{\overline{B}_R^c}(t,x,y)\ge c\Psi^2_{C_4,T}(t,|x-y|)$, where $\Psi^2_{a,T}(t,r)$ is defined in (1.4).*

Proof. By Lemma 4.3, we only need to prove the proposition for $|x-y|>\varphi^{-1}(t)/6$.

If $t/2\le\varphi(60R)$, then $\delta_{\overline{B}_R^c}(x)\wedge\delta_{\overline{B}_R^c}(y)\ge(M_1-1)R\ge 2R\ge(30)^{-1}\varphi^{-1}(t/2)$. In this case the Proposition holds by Theorem 3.8. So we only consider the following case: $t\ge T\wedge 2\varphi(60R)$ ($t\ge 2\varphi(60R)$ when $\beta=0$, respectively) and $|x-y|>\varphi^{-1}(t)/6$. Without loss of generality, we may assume that $|y|\ge|x-y|/2$. Let $x_1:=x+20^{-1}\varphi^{-1}(t/2)x/|x|$ then we have $B(x_1,20^{-1}\varphi^{-1}(t/2))\subset\overline{B}_{|x|}^c\subset\overline{B}_R^c$.

For every $z\in B(x_1,20^{-1}\varphi^{-1}(t/2))$, we obtain

$$|x-z|\le\frac{1}{20}\varphi^{-1}(t/2)+|x_1-z|\le\frac{1}{10}\varphi^{-1}(t/2)\le\frac{1}{6}\varphi^{-1}(t/2).\tag{4.6}$$

Since $|y| \geq |x-y|/2$ and $R \leq 60^{-1}\varphi^{-1}(t/2)$, we have

$$\delta_{\overline{B}_R^c}(y) = |y| - R \geq \frac{1}{2}|x-y| - \frac{1}{60}\varphi^{-1}(t/2) > \frac{1}{12}\varphi^{-1}(t) - \frac{1}{60}\varphi^{-1}(t/2) \geq \frac{1}{15}\varphi^{-1}(t/2). \tag{4.7}$$

For $z \in B(x_1, 60^{-1}\varphi^{-1}(t/2))$, we have

$$\begin{aligned}\delta_{\overline{B}_R^c}(z) = |z| - R &\geq |x_1| - |x_1 - z| - \frac{1}{60}\varphi^{-1}(t/2) \\ &\geq |x| + \frac{1}{20}\varphi^{-1}(t/2) - \frac{1}{60}\varphi^{-1}(t/2) - \frac{1}{60}\varphi^{-1}(t/2) \geq \frac{1}{60}\varphi^{-1}(t/2) \end{aligned} \tag{4.8}$$

and

$$|z-y| \leq |z-x| + |x-y| \leq \frac{1}{15}\varphi^{-1}(t/2) + |x-y| \leq 2|x-y|.$$

By the semigroup property, Lemma 4.3 with (4.6), Theorem 3.8 with (4.7) and (4.8) and the fact $r \to \Psi^2_{a,T}(t,r)$ is decreasing, there exist constants $c_i = c_i(\beta, T) > 0$ ($c_i > 0$ when $\beta = 0$, respectively), $i = 1, \dots, 4$ such that

$$\begin{aligned} p_{\overline{B}_R^c}(t,x,y) &= \int_{\overline{B}_R^c} p_{\overline{B}_R^c}(t/2,x,z) p_{\overline{B}_R^c}(t/2,z,y)dz \\ &\geq \int_{B(x_1,\varphi^{-1}(t/2)/60)} p_{\overline{B}_R^c}(t/2,x,z) p_{\overline{B}_R^c}(t/2,z,y)dz \\ &\geq c_1 \int_{B(x_1,\varphi^{-1}(t/2)/60)} 1/(\varphi^{-d}(t/2))\Psi^2_{c_2,T/2}(t/2,|z-y|)dz \\ &\geq c_3\, \Psi^2_{2c_2,T}(t, 2|x-y|) \geq\, c_4\, \Psi^2_{2^3c_2,T}(t,|x-y|). \end{aligned}$$

The last inequality holds by (1) in Lemma 2.2 with $a = 2c_2$ and $b = 2$ and we have proved the proposition. □

The following elementary lemma is used to prove the lower bound estimates on $p_D(t,x,y)$ where $t \in [T, \infty)$ ($t > 0$ when $\beta = 0$, respectively). Recall the function $\Psi^1_{a,b,T}(t,r)$ which is defined in (1.4).

Lemma 4.5 *Let K, R, b and t_0 be fixed positive constants and $\beta \in [0,\infty]$. Suppose that $x, x_1 \in \mathbb{R}^d$ satisfy $|x - x_1| = K^2R$. Then there exists a positive constant $c = c(K,R,b,t_0,\beta)$ such that for any $a > 0$ and $z \in \mathbb{R}^d$, we have $\Psi^1_{a,b,t_0}(t_0, 5|x-z|/4) \geq c\; \Psi^1_{a,b,t_0}(t_0, 2|x_1 - z|)$.*

Proof. Let $r := |x-z|$ and $r_1 := |x_1 - z|$. For any $z \in B(x, KR) \cup B(x_1, KR)$, we have that $r \leq (K+1)KR$. So $\Psi^1_{a,b,t_0}(t_0, 5r/4)$ is bounded below and the lemma holds.

Suppose that $z \notin B(x,KR) \cup B(x_1, KR)$. When $r \leq 4K^2R \vee 4/5$, then $\Psi^1_{a,b,t_0}(t_0, 5r/4)$ is bounded below and hence the lemma holds. Let $r > 4K^2R \vee 4/5$. By the triangle inequality, we have that $3r/4 < r - K^2R \leq r_1 \leq r + K^2R < 5r/4$ and hence $1 \leq 5r/4 \leq 5r_1/3 \leq 2r_1$. In this case, since $r \to \Psi^1_{a,b,t_0}(t_0, r)$ is non-increasing, the lemma holds. □

Now, we are ready to prove the lower bound estimates on $p_D(t,x,y)$. For the remainder of this paper, we assume that $\eta \in (\alpha/2, 1]$ and D is an exterior $C^{1,\eta}$ open set in $\mathbb{R}^d$ with $C^{1,\eta}$ characteristics (r_0, Λ_0) and $D^c \subset B(0,R)$ for some $R>0$. Such an open set D can be disconnected. When $\beta \in (1,\infty]$ and $|x-y| \ge 4/5$, we will consider the following two cases that x,y are in the same component and in different components in D, separately.

Proof of Theorem 1.2(2)–(3) Due to Theorem 1.1(4) and the domain monotonicity of $p_D(t,x,y)$, the Theorem holds when x,y are in the same bounded connected component of D. So we only need to prove Theorem 1.2(2)–(3.a).

When $\beta = 0$, by Theorem 1.1(1), we may assume that $t \ge T$. Without loss of generality, we may assume that $T=3$. For x and y in D, let $v \in \mathbb{R}^d$ be any unit vector satisfying $x \cdot v \ge 0$ and $y \cdot v \ge 0$. Let $M_2 := M_1(\beta, 3(16)^{-1}) (\ge 3)$, where M_1 is the constant in Lemma 4.2. Define

$$x_1 := x + M_2^2 R v \quad \text{and} \quad y_1 := y + M_2^2 R v.$$

By the semigroup property and Theorem 1.1(1)-(2), we have that for every $t-2 \ge 1$ and $x,y \in D$,

$$\begin{aligned} p_D(t,x,y) &= \int_D \int_D p_D(1,x,z) p_D(t-2,z,w) p_D(1,w,y) dz dw \\ &\ge c_1 (1 \wedge \delta_D(x))^{\alpha/2} (1 \wedge \delta_D(y))^{\alpha/2} f_2(t,x,y), \end{aligned} \tag{4.9}$$

where C_2 and γ are given constants in Theorem 1.1 and

$$\begin{aligned} f_2(t,x,y) = \int_{B(0,M_2R)^c \times B(0,M_2R)^c} &(1 \wedge \delta_D(z))^{\alpha/2} \Psi^1_{C_2,\gamma,1}(1, 5|x-z|/4) \\ &\cdot p_D(t-2,z,w)(1 \wedge \delta_D(w))^{\alpha/2} \Psi^1_{C_2,\gamma,1}(1, 5|y-w|/4) dz dw. \end{aligned} \tag{4.10}$$

Let $A_2 := \max\{(C_1C_4)^{1/(\beta\wedge 1)}, 2\gamma^{2/\beta}, 2C_1C_2\} (\ge 2)$ ($A_2 = 2$ when $\beta = 0$, respectively) where C_1 is the constant in (1.6), (1.7) and C_4 is the constant in Proposition 4.4. By Lemma 4.5 and (1.6), there exists $c_i = c_i(\beta) > 0, i = 2, \dots, 4$ such that

$$\begin{aligned} \Psi^1_{C_2,\gamma,1}(1, 5|x-z|/4) &\ge c_2 \Psi^1_{C_2,\gamma,1}(1, 2|x_1 - z|) \\ &\ge c_3 \Psi^1_{C_1^{-1},\gamma^{-1},1}(1, A_2|x_1-z|) \ge c_4 p(1, A_2x_1, A_2z) \quad \text{and} \\ \Psi^1_{C_2,\gamma,1}(1, 5|y-w|/4) &\ge c_2 \Psi^1_{C_2,\gamma,1}(1, 2|y_1 - w|) \\ &\ge c_3 \Psi^1_{C_1^{-1},\gamma^{-1},1}(1, A_2|y_1-w|) \ge c_4 p(1, A_2y_1, A_2w). \end{aligned} \tag{4.11}$$

When $\beta \in (0, \infty]$, the second inequalities hold by (2) in Lemma 2.1 along with $t_0 = 1$, $a = C_1$, $b = C_2$, $c = \gamma$, $N_1 = 2$ and $N_2 = A_2$ and the fact $A_2 \ge 2(C_1C_2 \vee \gamma^{2/\beta})$. When $\beta = 0$, the second inequalities hold since $A_2 = 2$.

For $z,w \in B(0,M_2R)^c$ and $t-2 \in [1,\infty)$, by Proposition 4.4 and (1.7), we have that

$$\begin{aligned} p_D(t-2,z,w) &\ge p_{\overline{B}_R^c}(t-2,z,w) \ge c_5 \Psi^2_{C_4,1}(t-2, |z-w|) \\ &\ge c_6 \Psi^2_{C_1^{-1},1}(t-2, A_2|z-w|) \ge c_7 p(t-2, A_2z, A_2w). \end{aligned} \tag{4.12}$$

For the third inequality above, we use (2) in Lemma 2.2 along with $T=1$, $a=C_4$, $b=C_1$ and $N=A_2$ and the fact $A_2\ge (C_1C_4)^{1/(\beta\wedge 1)}$ when $\beta\in(0,\infty]$. When $\beta=0$, the third inequality holds since $A_2\ge 1$.

For $z\in B(0,M_2R)^c$, $\delta_D(z)\ge \delta_{\overline{B}_R^c}(z)=|z|-R\ge M_2R-R$. So applying (4.11) and (4.12) to (4.10) and by the change of variables $\hat z=A_2z$, $\hat w=A_2w$ and semigroup property, we have that

$$
\begin{aligned}
f_2(t,x,y) &\ge c_8\int_{B(0,M_2R)^c\times B(0,M_2R)^c} p(1,A_2x_1,A_2z)p(t-2,A_2z,A_2w)p(1,A_2y_1,A_2w)dzdw\\
&\ge c_9\int_{B(0,A_2M_2R)^c\times B(0,A_2M_2R)^c} p_{B(0,A_2M_2R)^c}(1,A_2x_1,\hat z)p_{B(0,A_2M_2R)^c}(t-2,\hat z,\hat w)\\
&\qquad\qquad\cdot p_{B(0,A_2M_2R)^c}(1,A_2y_1,\hat w)d\hat z d\hat w\\
&= c_9\, p_{B(0,A_2M_2R)^c}(t,A_2x_1,A_2y_1). \qquad (4.13)
\end{aligned}
$$

Since $A_2|x_1|\wedge A_2|y_1|\ge M_2(A_2M_2R)$, by Proposition 4.4 and (1) in Lemma 2.2 with $a=C_4$ and $b=A_2$, we have that

$$
\begin{aligned}
p_{B(0,A_2M_2R)^c}(t,A_2x_1,A_2y_1) \ge& c_{10}\,\Psi^2_{C_4,T}(t,A_2|x_1-y_1|)\\
=& c_{10}\Psi^2_{C_4,T}(t,A_2|x-y|)\ge c_{11}\,\Psi^2_{A_2^2C_4,T}(t,|x-y|). \qquad (4.14)
\end{aligned}
$$

Combining (4.9) with (4.13) and (4.14), we have proved the lower bound estimates in Theorem 1.2(2)–(3.a). □

For the remainder of this section, we assume that $T>0$, $\beta\in(1,\infty)$ and $(t,x,y)\in[T,\infty)\times D\times D$ where $|x-y|\ge 4/5$ and x,y are in different components of D.

It is clear that there exists $0<\kappa\le 1/2$ which is depending on Λ_0 and d such that for all $x\in\overline{D}$ and $r\in(0,r_0]$ there is a ball $B(A_r(x),\kappa r)\subset D\cap B(x,r)$. Hereinafter, we assume that $A_r(x)$ is such the point in D.

Lemma 4.6 *Suppose that $D_b\subset B(0,R)$ be a bounded connected component of D. Then there exists a positive constant $c=c(\beta,\eta,r_0,\Lambda_0,T)$ such that for every $t\ge T$ and $x\in D_b$, we can find a ball $B\subset D_b$ such that*

$$
\int_B p_{D_b}(2^{-1}t-3^{-1}T,x,z)dz\ge c\,e^{-t\lambda^{D_b}}\delta_{D_b}(x)^{\alpha/2}
$$

where $-\lambda^{D_b}<0$ be the largest eigenvalue of the generator of Y^{D_b}.

Proof. For any $x\in D_b$, let $z_x\in\overline{D_b}$ be the point so that $|z_x-x|=\delta_{D_b}(x)$. Let $x_1:=A_{r_0}(z_x)$ and $B:=B(x_1,\kappa r_0)$. For any $z\in B$, we have that $\delta_{D_b}(z)\ge\kappa r_0$. Hence since $2^{-1}t-3^{-1}T\ge 6^{-1}T$, by Theorem 1.1(4) along with the bounded connected component D_b, there exist constants $c_i=c_i(\beta,\eta,r_0,\Lambda_0,T)>0$, $i=1,\dots,3$ such that for any $x\in D_b$

$$
\int_B p_{D_b}(2^{-1}t-3^{-1}T,x,z)dz\ge c_1e^{-t\lambda^{D_b}}\int_B\delta_{D_b}(x)^{\alpha/2}\delta_{D_b}(z)^{\alpha/2}dz
$$

$$\geq c_2 e^{-t\lambda^{D_b}} \delta_{D_b}(x)^{\alpha/2} \int_B dz \geq c_3 e^{-t\lambda^{D_b}} \delta_{D_b}(x)^{\alpha/2}.$$

$\square$

Now, we are ready to prove the lower bound estimates on $p_D(t,x,y)$ for any $\beta \in (1,\infty)$ and $(t,x,y) \in [T,\infty) \times D \times D$ where $|x-y| \geq 4/5$ and x,y are in different components of D.

Proof of Theorem 1.2(4) Let $D(x)$ and $D(y)$ be connected components containing x and y, respectively with $D(x) \cap D(y) \neq \emptyset$. Without loss of generality, we may assume that $D(x)$ is a bounded connected component and $T = 3$.

By the semigroup property and the domain monotonicity of $p_D(t,x,y)$, we first observe that

$$p_D(t,x,y) \geq \int_{D(x)} \int_{D(y)} p_{D(x)}(2^{-1}t-1,x,z) p_D(2,z,w) p_{D(y)}(2^{-1}t-1,y,w) dw dz. \tag{4.15}$$

For bounded connected component D_j of D and the largest eigenvalue $-\lambda_j < 0$ of the generator Y^{D_j}, define $\overline{\lambda} := \max\{\lambda_j : j = 1,\dots,n\}$. By Lemma 4.6, there exist a ball $B_x \subset D(x)$ and a constant $c_1 = c_1(\beta,\eta,r_0,\Lambda_0) > 0$ such that

$$\int_{B_x} p_{D(x)}(2^{-1}t-1,x,z) dz \geq c_1 e^{-t\overline{\lambda}} \delta_D(x)^{\alpha/2}. \tag{4.16}$$

Similarly, if $D(y)$ is a bounded connected component, we have that $\int_{B_y} p_{D(y)}(2^{-1}t-1,y,w)dw \geq c_2 e^{-t\overline{\lambda}} \delta_D(y)^{\alpha/2}$ for some a ball $B_y \subset D(y)$ and a constant $c_2 > 0$. For any $(z,w) \in B_x \times B_y$, note that $r_0 \leq |z-w| \leq 2R$ and $\delta_D(z) \wedge \delta_D(w) \geq c_3$. So by Theorem 1.1(1) and (3), we have that $\inf_{(z,w)\in B_x\times B_y} p_D(2,z,w) \geq c_4$. Hence, we have the conclusion when $D(x)$ and $D(y)$ are bounded connected components of D.

When $D(y)$ is an unbounded connected component, let $y_1 := y + 2Ry/|y|$ and $B_{y_1} := B(y_1, 2^{-1}R) \subset D(y)$. For any $w \in B_{y_1}$, we have that $\delta_{D(y)}(w) \geq R/2$ and $|y-w| \leq |y-y_1| + |y_1-w| \leq 5R/2$. Hence for $2^{-1}t - 1 \geq 1/2$, by Theorem 1.2(2)–(3.a) and the fact $t/2 - 1 \asymp t$, there exist constants $c_i = c_i(\beta,\eta,r_0,\Lambda_0,R) > 0$, $i = 5,\dots,8$ such that

$$\begin{aligned}\int_{B_{y_1}} p_{D(y)}(2^{-1}t-1,y,w)dw &\geq c_5 \int_{B_{y_1}} (1\wedge\delta_{D(y)}(y))^{\alpha/2} (1\wedge\delta_{D(y)}(w))^{\alpha/2} t^{-d/2} \exp(-c_6|y-w|^2/t) dw \\ &\geq c_7 (1\wedge\delta_D(y))^{\alpha/2} t^{-d/2} \int_{B_{y_1}} dw = c_8 (1\wedge\delta_D(y))^{\alpha/2} t^{-d/2}. \end{aligned} \tag{4.17}$$

For any $(z,w) \in B_x \times B_{y_1}$, we have that $\delta_D(z) \wedge \delta_D(w) \geq c_9$ and

$$|z-w| \leq |z-x| + |x-y| + |y-w| \leq 2R + |x-y| + 5R/2 \leq c_{10}|x-y|.$$

The last inequality holds since $|x-y| \geq 4/5$. So by Theorem 1.1(1) and (3), there are constants $c_i = c_i(\beta,\eta,r_0,\Lambda_0,R) > 0$, $i = 11,\dots,14$ such that

$$\inf_{(z,w)\in B_x\times B_{y_1}} p_D(2,z,w) \geq c_{11} \left(\frac{\exp(-c_{12}|z-w|^\beta)}{|z-w|^{d+\alpha}} \wedge 1 \right) \geq c_{13} \frac{\exp(-c_{14}|x-y|^\beta)}{|x-y|^{d+\alpha}}. \tag{4.18}$$

Combining (4.16), (4.17) and (4.18) with (4.15), we have the conclusion when $D(x)$ is a bounded connected component and $D(y)$ is an unbounded connected component of D. $\Box$

Remark 4.7 *Let D be an exterior $C^{1,\eta}$ open set in $\mathbb{R}^d$ with $C^{1,\eta}$ characteristics (r_0, Λ_0) and $D^c \subset B(0,R)$. Then the number of bounded connected components of D is finite, say $D_1, \ldots, D_n$. According to the proof of Theorem 1.2(4), there exists a constant $c > 0$ such that if $x, y \in D$ are in different bounded connected components of D*

$$p_D(t,x,y) \geq c\,\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} \exp\left(-t\sum_{j=1}^{n}\lambda_j\Big(\mathbf{1}_{D_j}(x) + \mathbf{1}_{D_j}(y)\Big)\right)$$

where $-\lambda_j < 0$ is the largest eigenvalue of the generator Y^{D_j}, $j = 1, \cdots, n$.

5 Green function estimate

In this section, we present a proof of Corollary 1.4. We recall that $G_D(x,y) = \int_0^\infty p_D(t,x,y)dt$. When $\beta = 0$, the proof of Corollary 1.4 is similar to that of [16, Corollary 1.5], we only consider the case $\beta \in (0, \infty]$.

Proof of Corollary 1.4 By Corollary 1.3, there exist constants $c_i > 1$, $i = 1, 2$ such that

$$\begin{aligned} G_D(x,y) &\leq c_1 \int_0^1 \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \Psi^1_{c_2^{-1},\gamma^{-1},30}(t, |x-y|/6)dt \\ &\quad + c_1 (1 \wedge \delta_D(x))^{\alpha/2} (1 \wedge \delta_D(y))^{\alpha/2} \int_1^\infty \Psi^2_{c_2^{-1},1}(t, |x-y|)dt \qquad \text{and} \\ G_D(x,y) &\geq c_1^{-1} \cdot \mathbf{1}_{\{|x-y|<4/5\}} \int_0^1 \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \Psi^1_{c_2,\gamma,30}(t, 5|x-y|/4)dt \\ &\quad + c_1^{-1} \cdot \mathbf{1}_{\{|x-y|\geq 4/5\}} (1 \wedge \delta_D(x))^{\alpha/2} (1 \wedge \delta_D(y))^{\alpha/2} \int_1^\infty \Psi^2_{c_2,1}(t, |x-y|)dt \end{aligned}$$

where γ is the constant in Theorem 1.1.

Without loss of generality, we may assume that $c_2 = 1$ and we define I_1, I_2 and II by

$$\begin{aligned} I_1 &:= \int_0^1 \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \left(t^{-d/\alpha} \wedge t|x-y|^{-\alpha-d}\right) dt \\ I_2 &:= \int_0^1 \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha/2} \Psi^1_{1,\gamma^{-1},30}(t, |x-y|/6)dt \\ II &:= \int_1^\infty \Psi^2_{1,1}(t, |x-y|)dt. \end{aligned}$$

For any $a, b > 0$, if $b|x-y| < 1$, we have that $\Psi^1_{1,a,30}(t, b|x-y|) \asymp t^{-d/\alpha} \wedge t|x-y|^{-\alpha-d}$. So when $|x-y| < 4/5$, it suffices to show that

$$I_1 \asymp \left(\frac{1}{|x-y|^{d-\alpha}} + \frac{1}{|x-y|^{d-2}}\right) \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)^{\alpha/2} \quad \text{and}$$

$$II \leq c_3 \leq c_4 \left(\frac{1}{|x-y|^{d-\alpha}} + \frac{1}{|x-y|^{d-2}} \right). \tag{5.1}$$

When $|x-y| \geq 4/5$, we will show that

$$I_2 \leq c_5 \left(\frac{1}{|x-y|^{d-\alpha}} + \frac{1}{|x-y|^{d-2}} \right) (1 \wedge \delta_D(x))^{\alpha/2} (1 \wedge \delta_D(y))^{\alpha/2} \text{ and} \tag{5.2}$$

$$II \asymp \frac{1}{|x-y|^{d-2}} \asymp \left(\frac{1}{|x-y|^{d-\alpha}} + \frac{1}{|x-y|^{d-2}} \right). \tag{5.3}$$

Let $r := |x-y|$. Suppose that $r < 4/5$. By [[7], (4.3), (4.4) and (4.6)], we have

$$\begin{aligned} I_1 &\asymp \frac{1}{r^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)}{r} \right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{r} \right)^{\alpha/2} \\ &\asymp \left(\frac{1}{r^{d-\alpha}} + \frac{1}{r^{d-2}} \right) \left(1 \wedge \frac{\delta_D(x)}{r} \right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{r} \right)^{\alpha/2}. \end{aligned} \tag{5.4}$$

Note that for every $s \in [0, \infty]$,

$$\int_s^\infty t^{-d/2} e^{-r^2/t} dt = r^{2-d} \int_0^{r^2/s} u^{d/2-2} e^{-u} du. \tag{5.5}$$

For $r < 1$ and $1 < t$, we have $\Psi^2_{1,1}(t,r) = t^{-d/2} e^{-r^2/t}$ and

$$II = r^{2-d} \int_0^{r^2} u^{d/2-2} e^{-u} du \asymp r^{2-d} \int_0^{r^2} u^{d/2-2} du = \frac{2}{d-2}. \tag{5.6}$$

Hence we obtain (5.1) by (5.4) and (5.6).

Suppose that $r \geq 4/5$. Note that for $0 < t \leq 1$, we have

$$\begin{aligned} \Psi^1_{1,\gamma^{-1},30}(t, r/6) &= \begin{cases} t^{-d/\alpha} \wedge t(r/6)^{-d-\alpha} e^{-\gamma^{-1}(r/6)^\beta} & \leq t(r/6)^{-d-\alpha} \quad \text{for } \beta \in (0,1] \\ t \exp(-((r/6)(\log(5r/t))^{(\beta-1)/\beta} \wedge (r/6)^\beta)) & \leq te^{-c_6 r} \quad \text{for } \beta \in (1,\infty) \\ (t/(5r))^{r/6} & \leq t^{2/15} e^{-c_6 r} \quad \text{for } \beta = \infty \end{cases} \\ &\leq c_7 t^{2/15} r^{-d-\alpha} \end{aligned}$$

for some constant $c_i = c_i(\beta) > 0$, $i = 6, 7$. Thus by the change of variable $u = r^\alpha/t$, there exist constants $c_i > 0, i = 8, 9$ such that

$$\begin{aligned} I_2 &\leq c_7 r^{-d-\alpha} \int_0^1 t^{\frac{2}{15}} \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}} \right)^{\alpha/2} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}} \right)^{\alpha/2} dt \\ &= c_7 r^{-d+\frac{2}{15}\alpha} \int_{r^\alpha}^\infty u^{-\frac{2}{15}-2} \left(1 \wedge \frac{\sqrt{u}\delta_D(x)^{\alpha/2}}{r^{\alpha/2}} \right) \left(1 \wedge \frac{\sqrt{u}\delta_D(y)^{\alpha/2}}{r^{\alpha/2}} \right) du \\ &= c_7 r^{-d+\frac{2}{15}\alpha} \int_{r^\alpha}^\infty u^{-\frac{2}{15}-1} \left(\frac{1}{\sqrt{u}} \wedge \frac{\delta_D(x)^{\alpha/2}}{r^{\alpha/2}} \right) \left(\frac{1}{\sqrt{u}} \wedge \frac{\delta_D(y)^{\alpha/2}}{r^{\alpha/2}} \right) du \end{aligned}$$

$$\le c_8 r^{-d+\frac{2}{15}\alpha}\int_{r^\alpha}^\infty u^{-\frac{2}{15}-1}du\left(1\wedge\frac{\delta_D(x)}{r}\right)^{\alpha/2}\left(1\wedge\frac{\delta_D(y)}{r}\right)^{\alpha/2}$$
$$=\frac{15}{2}c_8 r^{-d}\left(1\wedge\frac{\delta_D(x)}{r}\right)^{\alpha/2}\left(1\wedge\frac{\delta_D(y)}{r}\right)^{\alpha/2}\le\ c_9 r^{2-d}\left(1\wedge\delta_D(x)\right)^{\alpha/2}\left(1\wedge\delta_D(y)\right)^{\alpha/2} \tag{5.7}$$

and it yields (5.2). For (5.3), because of (5.6), we may assume that $r\ge 1$. By (5.5), we have that

$$II\ge\int_1^\infty t^{-d/2}e^{-r^2/t}dt\ge r^{2-d}\int_0^1 u^{d/2-2}e^{-u}du\ge c_{10}r^{2-d}\qquad\text{and}$$
$$II\le\int_1^{r^{2-(\beta\wedge1)}} t^{-d/2}e^{-r^{(\beta\wedge1)}}dt+\int_{r^{2-(\beta\wedge1)}}^\infty t^{-d/2}e^{-r^2/t}dt$$
$$\le c_{11}e^{-r^{(\beta\wedge1)}}+r^{2-d}\int_0^{r^{(\beta\wedge1)}}u^{d/2-2}e^{-u}du\le c_{12}r^{2-d}.$$

This implies $II\asymp r^{2-d}$ and hence (5.3) holds. So we have proved the Corollary. □

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